Lexicographic Shellability Statistics

Alexander Lazar

KTH

June 2022

Joint with Joseph Doolittle (TU Graz) and Bennet Goeckner (U. San Diego)

Preliminaries: Matroids and Shellability

Throughout, all simplicial complexes are d-dimensional with n vertices.

Preliminaries: Matroids and Shellability

Throughout, all simplicial complexes are d-dimensional with n vertices.

Ordered simplicial complex $\Delta_{\prec}$: simplicial complex Δ along with an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of its vertices.

Preliminaries: Matroids and Shellability

Throughout, all simplicial complexes are d-dimensional with n vertices.

Ordered simplicial complex $\Delta_\prec$: simplicial complex Δ along with an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of its vertices.

Faces of $\Delta_\prec$ are ordered sets: $F = \{v_{i_1} \prec \cdots \prec v_{i_{k+1}}\}$.

Preliminaries: Matroids and Shellability

Throughout, all simplicial complexes are d-dimensional with n vertices.

Ordered simplicial complex $\Delta_{\prec}$: simplicial complex Δ along with an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of its vertices.

Faces of $\Delta_{\prec}$ are ordered sets: $F = \{v_{i_1} \prec \cdots \prec v_{i_{k+1}}\}$.

Theorem (Björner)

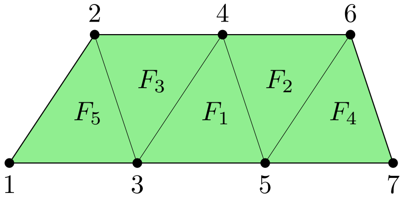
Let Δ be a pure simplicial complex. Then Δ is the independence complex of a matroid if and only if, for every order $\prec$ of the vertices of Δ, the lexicographic order of the facets of $\Delta_{\prec}$ is a shelling order.

Preliminaries: Matroids and Shellability

Shelling order: A total order $F_1, \ldots, F_k$ of the facets of Δ such that $F_j \setminus (F_1 \cup \cdots \cup F_{j-1})$ has a unique minimal element for all $j > 1$.

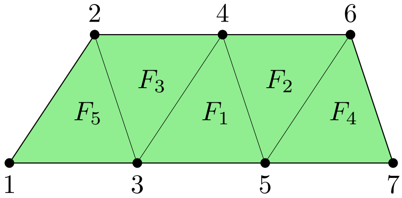
Preliminaries: Matroids and Shellability

Shelling order: A total order $F_1, \ldots, F_k$ of the facets of Δ such that $F_j \setminus (F_1 \cup \cdots \cup F_{j-1})$ has a unique minimal element for all $j > 1$.



Preliminaries: Matroids and Shellability

Shelling order: A total order $F_1, \ldots, F_k$ of the facets of Δ such that $F_j \setminus (F_1 \cup \cdots \cup F_{j-1})$ has a unique minimal element for all $j > 1$.



Question: What can we say about complexes where some, but maybe not all, total vertex orders make lex-order into a shelling order?

Motivation

Motivation

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.
  - Coleman–Dochtermann–Geist–Oh: revlex order is a shelling order $\implies$ shelling of Δ can be extended to shelling of the d-skeleton of the $(n-1)$-simplex.

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.
  - Coleman–Dochtermann–Geist–Oh: revlex order is a shelling order $\implies$ shelling of Δ can be extended to shelling of the d-skeleton of the $(n-1)$-simplex.
- Quasi-matroidal classes (Samper)

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.
    - Coleman–Dochtermann–Geist–Oh: revlex order is a shelling order $\Longrightarrow$ shelling of Δ can be extended to shelling of the d-skeleton of the $(n-1)$-simplex.
- Quasi-matroidal classes (Samper)
- Random graph theory

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.
  - Coleman–Dochtermann–Geist–Oh: revlex order is a shelling order $\implies$ shelling of Δ can be extended to shelling of the d-skeleton of the $(n-1)$-simplex.
- Quasi-matroidal classes (Samper)
- Random graph theory
- $\frac{5}{8}$ theorem:

Motivation

- Simon's conjecture: The d-skeleton of the $(n-1)$-simplex is extendably shellable.
  - Coleman–Dochtermann–Geist–Oh: revlex order is a shelling order $\Longrightarrow$ shelling of Δ can be extended to shelling of the d-skeleton of the $(n-1)$-simplex.
- Quasi-matroidal classes (Samper)
- Random graph theory
- $\frac{5}{8}$ theorem:
  - If the probability that a pair of elements (chosen uniformly at random) in a finite group G commutes is $> \frac{5}{8}$, then G is abelian.

The Lex-Shellability Statistic

An order $\prec$ on Δ is shelling-compatible if the $\prec$-lexicographic order on the facets of Δ is a shelling order.

The Lex-Shellability Statistic

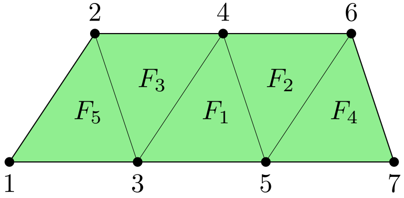
An order $\prec$ on Δ is shelling-compatible if the $\prec$-lexicographic order on the facets of Δ is a shelling order.

$\mathfrak{L}(\Delta) := \frac{1}{n!} \cdot \#\{\text{s.c. vertex orders on } \Delta\}$

The Lex-Shellability Statistic

An order $\prec$ on Δ is <u>shelling-compatible</u> if the $\prec$-lexicographic order on the facets of Δ is a shelling order.

$\mathfrak{L}(\Delta) := \frac{1}{n!} \cdot \#\{\text{s.c. vertex orders on } \Delta\}$



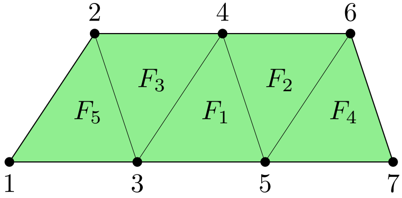
1234567 ✓ 1235674 × 4536271 ✓

$\mathfrak{L}(\Delta) = \frac{1}{3}$

The Lex-Shellability Statistic

An order $\prec$ on Δ is shelling-compatible if the $\prec$-lexicographic order on the facets of Δ is a shelling order.

$\mathfrak{L}(\Delta) := \frac{1}{n!} \cdot \#\{\text{s.c. vertex orders on } \Delta\}$



1234567 ✓ 1235674 × 4536271 ✓

$$\mathfrak{L}(\Delta) = \frac{1}{3}$$

Björner's theorem: Δ is the independence complex of a matroid iff $\mathfrak{L}(\Delta) = 1$.

Properties of $\mathfrak{L}(\Delta)$

Theorem (Doolittle–Goeckner–L.)

1. $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$, *where* $*$ *is simplicial join.*

Properties of $\mathfrak{L}(\Delta)$

Theorem (Doolittle–Goeckner–L.)

1. $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$, *where* $*$ *is simplicial join.*
2. *For any* $\epsilon > 0$ *there exist complexes with* $0 < \mathfrak{L}(\Delta) < \epsilon$ *and complexes with* $1 - \epsilon < \mathfrak{L}(\Delta) < 1$.

Properties of $\mathfrak{L}(\Delta)$

Theorem (Doolittle–Goeckner–L.)

1. $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$, *where* $*$ *is simplicial join.*
2. *For any* $\epsilon > 0$ *there exist complexes with* $0 < \mathfrak{L}(\Delta) < \epsilon$ *and complexes with* $1 - \epsilon < \mathfrak{L}(\Delta) < 1$.
3. *There exist shellable (in fact, vertex-decomposable) complexes with* $\mathfrak{L}(\Delta) = 0$.

Properties of $\mathfrak{L}(\Delta)$

Theorem (Doolittle–Goeckner–L.)

1. $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$, *where* $*$ *is simplicial join.*
2. *For any* $\epsilon > 0$ *there exist complexes with* $0 < \mathfrak{L}(\Delta) < \epsilon$ *and complexes with* $1 - \epsilon < \mathfrak{L}(\Delta) < 1$.
3. *There exist shellable (in fact, vertex-decomposable) complexes with* $\mathfrak{L}(\Delta) = 0$.
4. *There exist complexes which are not vertex-decomposable, but for which* $\mathfrak{L}(\Delta) > 0$.

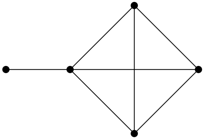
Properties of $\mathfrak{L}(\Delta)$

Theorem (Doolittle–Goeckner–L.)

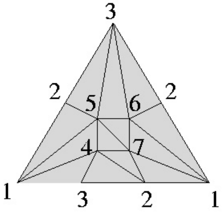
1. $\mathfrak{L}(\Delta_1 * \Delta_2) = \mathfrak{L}(\Delta_1)\mathfrak{L}(\Delta_2)$, *where* $*$ *is simplicial join.*
2. *For any* $\epsilon > 0$ *there exist complexes with* $0 < \mathfrak{L}(\Delta) < \epsilon$ *and complexes with* $1 - \epsilon < \mathfrak{L}(\Delta) < 1$.
3. *There exist shellable (in fact, vertex-decomposable) complexes with* $\mathfrak{L}(\Delta) = 0$.
4. *There exist complexes which are not vertex-decomposable, but for which* $\mathfrak{L}(\Delta) > 0$.
5. *If* $\mathfrak{L}(\Delta) > 0$, *the face poset of* Δ *is EL-shellable.*

Proof Sketch(es)

$$\lim_{n\to\infty} \mathfrak{L}(P_n) = 0$$



$$\lim_{n\to\infty} \mathfrak{L}(\overline{K_n}) = 1$$



Hachimori's complex: not v.d. but $\mathfrak{L} > 0$
`http://infoshako.sk.tsukuba.ac.jp/~hachi/math/library/nonextend_eng.html`

Random Complexes

Q: Suppose I select a d-complex Δ with vertex set $[n]$ uniformly at random. What's $\mathrm{E}(\mathfrak{L}(\Delta))$?

Note:

$$\begin{aligned}\mathrm{E}(\mathfrak{L}(\Delta)) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{E}\left(\mathbb{1}(\sigma \text{ is s.c. for uniformly chosen } \Delta)\right) \\ &= \mathrm{P}(\mathrm{id} \text{ is s.c. for uniformly chosen } \Delta)\end{aligned}$$

Random Complexes

Q: Suppose I select a d-complex Δ with vertex set $[n]$ uniformly at random. What's $\mathrm{E}(\mathfrak{L}(\Delta))$?

Note:

$$\begin{aligned}\mathrm{E}(\mathfrak{L}(\Delta)) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{E}\left(\mathbb{1}(\sigma \text{ is s.c. for uniformly chosen } \Delta)\right) \\ &= \mathrm{P}(\mathrm{id} \text{ is s.c. for uniformly chosen } \Delta)\end{aligned}$$

In general, seems difficult (some experimental data for $d = 2$ and $n \leq 6$).

Random Graphs

Fact. Let $\mathcal{G}_n$ be "the" uniformly-chosen graph on $[n]$ (a.k.a. Erdös–Rényi graph with $p = \frac{1}{2}$). Then

$$\lim_{n \to \infty} \mathrm{P}(\mathrm{diam}(\mathcal{G}_n) \leq 2) = 1.$$

Random Graphs

Fact. Let $\mathcal{G}_n$ be "the" uniformly-chosen graph on $[n]$ (a.k.a. Erdös–Rényi graph with $p = \frac{1}{2}$). Then

$$\lim_{n\to\infty} \mathrm{P}(\mathrm{diam}(\mathcal{G}_n) \leq 2) = 1.$$

In particular, we then have

$$\lim_{n\to\infty} \mathrm{P}(\mathrm{id} \text{ is s.c. for } \mathcal{G}_n) = 1,$$

Random Graphs

Fact. Let $\mathcal{G}_n$ be "the" uniformly-chosen graph on $[n]$ (a.k.a. Erdös–Rényi graph with $p = \frac{1}{2}$). Then

$$\lim_{n\to\infty} \mathrm{P}(\mathrm{diam}(\mathcal{G}_n) \leq 2) = 1.$$

In particular, we then have

$$\lim_{n\to\infty} \mathrm{P}(\mathrm{id} \text{ is s.c. for } \mathcal{G}_n) = 1,$$

so

Proposition (Doolittle–Goeckner–L.)

$$\lim_{n\to\infty} \mathrm{E}(\mathfrak{L}(\mathcal{G}_n)) = 1.$$

Quasi-Matroidal Classes

Definition (Samper)

A class $\mathcal{A}$ of ordered pure simplicial complexes is a quasi-matroidal class if

1. Every ordered matroid belongs to $\mathcal{A}$
2. If an ordered complex $\Delta_{\prec}$ belongs to $\mathcal{A}$ and every other ordering $\prec'$ of the vertices of Δ yields an ordered complex $\Delta_{\prec'}$ in $\mathcal{A}$, then Δ is a matroid independence complex
3. Every pure shifted complex is in $\mathcal{A}$
4. $\mathcal{A}$ is closed under the following operations:
   1. **Join:** If $\Delta_{\prec}$ and $\Gamma_{\prec'}$ are in $\mathcal{A}$, then $(\Delta * \Gamma)_{\prec''}$ is in $\mathcal{A}$, where $\prec''$ is any shuffle of the vertex orders $\prec$ and $\prec'$
   2. **Deletion:** If $\Delta_{\prec}$ is in $\mathcal{A}$ and i_n is the largest vertex of Δ with respect to $\prec$, then $(\mathrm{del}_{\Delta}(i_n))_{\prec}$ is in $\mathcal{A}$
   3. **Contraction:** If $\Delta_{\prec}$ is in $\mathcal{A}$, then $(\mathrm{lk}_{\Delta}(\sigma))_{\prec}$ is also in $\mathcal{A}$ for all $\sigma \in \Delta$.

LEX and PURE

LEX: largest class of $\Delta_{\prec}$ where $\prec$ is s.c. for Δ that is closed under joins, deletions, and contractions.

PURE: $\Delta_{\prec} \in \mathrm{PURE}$ iff

- Δ is a simplex, or
- $(\Delta \setminus \{v\})_{\prec} \in \mathrm{PURE}$ of the same dimension for v largest vertex and $\mathrm{lk}_{\Delta}(v)_{\prec} \in \mathrm{PURE}$ for all v.

Proposition (Doolittle–Goeckner–L.)

- $\Delta_{\prec} \in \mathrm{PURE} \implies \prec$ *s.c. for* Δ.
- $\Delta_{\prec} \in \mathrm{LEX} \implies \prec$ *s.c. for* Δ.
- $\prec$ *s.c. for* $\Delta \not\Longrightarrow \Delta_{\prec} \in \mathrm{LEX}$.

Current Directions

- Analogs of $\mathfrak{L}$ for other matroid axioms?
- Nontrivial thresholds for $\mathfrak{L}(\Delta)$?
- $\mathfrak{L}(\Delta) > 0 \implies$ shellings of Δ extend to d-skeleton of $(n-1)$-simplex?